The ratio and generating function of cogrowth coefficients of finitely generated groups

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Abstract

Let G be a group generated by r elements $g_1, g_2, \ldots, g_r$. Among the reduced words in $g_1, g_2, \ldots, g_r$ of length n some, say γ_n, represent the identity element of the group G. It has been shown in a combinatorial way that the $2n$th root of γ_{2n} has a limit, called the cogrowth exponent with respect to generators $g_1, g_2, \ldots, g_r$. We show by analytic methods that the numbers γ_n vary regularly; i.e. the ratio $\gamma_{2n+2}/\gamma_{2n}$ is also convergent. Moreover we derive new precise information on the domain of holomorphy of $\gamma(z)$, the generating function associated with the coefficients γ_n.

Every group G generated by r elements can be realized as a quotient of the free group $\mathbb{F}_r$ on r generators by a normal subgroup N of $\mathbb{F}_r$, in such a way that the generators of the free group $\mathbb{F}_r$ are sent to the generators of the group G. With the set of generators of $\mathbb{F}_r$ we associate the length function of words in these generators. The cogrowth coefficients $\gamma_n = \#\{x \in N \mid |x| = n\}$ were first introduced by Grigorchuk in [2]. The numbers γ_n measure how big the group G is when compared with $\mathbb{F}_r$. It has been shown that the quantities $\sqrt[2n]{\gamma_{2n}}$ have a limit denoted by γ, and called the growth exponent of N in $\mathbb{F}_r$. Since the subgroup N can have at most $2r(2r-1)^{n-1}$ elements of length n, the cogrowth exponent γ can be at most $2r-1$. The famous Grigorchuk result, proved independently by J. M. Cohen in [1], states that the group G is amenable if and only if $\gamma = 2r - 1$ (see also [6], [8]).

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The main result of this note is that the coefficients γ_{2n} satisfy not only the Cauchy nth root test but also the d'Alambert ratio test.

Theorem 1 *The ratio of two consecutive even cogrowth coefficients $\gamma_{2n+2}/\gamma_{2n}$ has a limit. Thus the ratio tends to γ^2, the square of the cogrowth exponent.*

Proof. Let us denote by $g_1, g_2, \ldots, g_r$ the generators of G. Let μ be the measure equidistributed over the generators and their inverses according to the formula

$$\mu = \frac{1}{2\sqrt{q}} \sum_{i=1}^{r} (g_i + g_i^{-1}),$$

where $q = 2r - 1$. By an easy transformation of [6, Formula (*)] we obtain

$$\frac{z}{1-z^2} \sum_{n=0}^{\infty} \gamma_n z^n = \frac{1}{2\sqrt{q}} \sum_{n=0}^{\infty} \mu^{*n}(e) \left(\frac{2\sqrt{q}z}{qz^2+1} \right)^{n+1}, \tag{1}$$

for small values of $|z|$. Let ϱ denote the spectral radius of the random walk defined by μ; i.e.

$$\varrho = \lim_{n\to\infty} \sqrt[2n]{\mu^{*2n}(e)}.$$

By $d\sigma(x)$ we will denote the spectral measure of this random walk. Hence

$$\mu^{*n}(e) = \int_{-\varrho}^{\varrho} x^n d\sigma(x). \tag{2}$$

Note that the point ϱ belongs to the support of σ. Combining (1) and (2) gives

$$\begin{aligned} \frac{z}{1-z^2} \sum_{n=0}^{\infty} \gamma_n z^n &= \frac{1}{2\sqrt{q}} \int_{-\varrho}^{\varrho} \sum_{n=0}^{\infty} x^n \left(\frac{2\sqrt{q}z}{qz^2+1} \right)^{n+1} d\sigma(x) \\ &= \frac{1}{2\sqrt{q}} \int_{-\varrho}^{\varrho} \frac{z}{1-2\sqrt{q}xz+qz^2} d\sigma(x). \end{aligned} \tag{3}$$

By the well known formula for the generating function of the second kind Chebyshev polynomials $U_n(x)$ (see [4, (4.7.23), page 82]) where

$$U_n\left(\tfrac{1}{2}(t+t^{-1})\right) = \frac{t^{n+1} - t^{-n-1}}{t - t^{-1}}, \tag{4}$$

we have

$$\frac{1}{1-2\sqrt{q}xz+qz^2} = \sum_{n=0}^{\infty} U_n(x) q^{n/2} z^n.$$

Thus

$$\frac{z}{1-z^2}\sum_{n=0}^{\infty}\gamma_n z^n = z\sum_{n=0}^{\infty} q^{n/2} z^n \int_{-\varrho}^{\varrho} U_n(x)\,d\sigma(x).$$

Therefore for $n \geq 2$ we have

$$\gamma_n = q^{n/2}\int_{-\varrho}^{\varrho}\left\{U_n(x) - q^{-1}U_{n-2}(x)\right\} d\sigma(x). \tag{5}$$

Since $U_{2n}(-x) = U_{2n}(x)$ we get

$$\gamma_{2n} = q^n \int_0^{\varrho}\left\{U_{2n}(x) - q^{-1}U_{2n-2}(x)\right\} d\widetilde{\sigma}(x), \tag{6}$$

where $\widetilde{\sigma}(A) = \sigma(A) + \sigma(-A)$ for $A \subset (0,\varrho]$ and $\widetilde{\sigma}(\{0\}) = \sigma(\{0\})$. Let

$$I_n = \int_0^{\varrho}\left\{U_{2n}(x) - q^{-1}U_{2n-2}(x)\right\} d\widetilde{\sigma}(x).$$

By [3, Corollary 2] we have $\varrho > 1$. Hence we can split the integral I_n into two integrals: the first $I_{n,1}$ over the interval $[0,\varrho_0]$ and the second $I_{n,2}$ over $[\varrho_0,\varrho]$, where $\varrho_0 = (1+\varrho)/2$. By (4) we have $|U_m(x)| \leq (m+1)$ for $x \in [0,1]$ and

$$|U_m(x)| \leq (m+1)[x+\sqrt{x^2-1}]^m \quad \text{for } x \geq 1.$$

Thus we get

$$\begin{aligned} I_{n,1} &\leq 2(2n+1)\left(\varrho_0 + \sqrt{\varrho_0^2-1}\right)^{2n}\int_0^{\varrho_0} d\widetilde{\sigma}(x) \\ &\leq 2(2n+1)\left(\varrho_0 + \sqrt{\varrho_0^2-1}\right)^{2n}. \end{aligned} \tag{7}$$

Let's turn to estimating the integral $I_{n,2}$ over $[\varrho_0,\varrho]$. By (4) one can easily check that

$$\left|U_n(x) - \frac{(x+\sqrt{x^2-1})^{n+1}}{2\sqrt{x^2-1}}\right| = o(1) \quad \text{when } n \to \infty,$$

uniformly on the interval $[\varrho_0, \varrho]$. Hence

$$\left| U_{2n}(x) - q^{-1}U_{2n-2}(x) - (x+\sqrt{x^2-1})^{2n-1}\frac{(x+\sqrt{x^2-1})^2 - q^{-1}}{2\sqrt{x^2-1}} \right| = o(1),$$

when n tends to infinity, uniformly in the interval $[\varrho_0, \varrho]$. This implies

$$I_{n,2} \approx \widetilde{I}_{n,2} = \int_{\varrho_0}^{\varrho} (x+\sqrt{x^2-1})^{2n} \frac{(x+\sqrt{x^2-1})^2 - q^{-1}}{2\sqrt{x^2-1}(x+\sqrt{x^2-1})}\, d\widetilde{\sigma}(x). \qquad (8)$$

Since the endpoint ϱ belongs to the support of $\widetilde{\sigma}$, we get

$$\widetilde{I}_{n,2}^{1/2n} \longrightarrow \varrho + \sqrt{\varrho^2 - 1}. \qquad (9)$$

By combining this with (7) and (8) we obtain

$$I_n = I_{n,1} + I_{n,2} = \widetilde{I}_{n,2}(1 + o(1)), \qquad n \to \infty. \qquad (10)$$

In view of (9) the integral $\widetilde{I}_{n,2}$ tends to infinity. Thus by (6) and (10) we have

$$\frac{\gamma_{2n+2}}{\gamma_{2n}} \approx q \frac{\widetilde{I}_{n+1,2}}{\widetilde{I}_{n,2}}.$$

Lemma 1 ([7]) *Let $f(x)$ be a positive and continuous function on $[a,b]$, and μ be a finite measure on $[a,b]$. Then*

$$\lim_{n\to\infty} \frac{\int_a^b f(x)^{n+1} d\mu(x)}{\int_a^b f(x)^n d\mu(x)} = \max\{f(x) \mid x \in \operatorname{supp}\mu\}.$$

Applying Lemma 1 and using the fact that ϱ belongs to the support of $\widetilde{\sigma}$ gives

$$\frac{\gamma_{2n+2}}{\gamma_{2n}} \to q\left\{\varrho + \sqrt{\varrho^2-1}\right\}^2. \qquad (11)$$

$\square$

Theorem 2 *The generating function $\gamma(z) = \sum_{n=0}^{\infty} \gamma_n z^n$ can be decomposed into a sum of two functions $\gamma^{(0)}(z)$ and $\gamma^{(1)}(z)$ such that $\gamma^{(0)}(z)$ is analytic in the open disc of radius $q^{-1/2}$ (where $q = 2r-1$), while $\gamma^{(1)}(z)$ is analytic in the whole complex plane after removing the two real intervals $[-\gamma q^{-1}, -\gamma^{-1}]$ and $[\gamma^{-1}, \gamma q^{-1}]$. Moreover, $\gamma^{(1)}$ satisfies the functional equation*

$$\frac{z\gamma^{(1)}(z)}{1-z^2} = \frac{(q/z)\gamma^{(1)}(q/z)}{(q/z)}.$$

Proof. By (3) we have

$$\gamma(z) = (1-z^2)\int_{-\varrho}^{\varrho} \frac{1}{1-2\sqrt{q}xz+qz^2} d\sigma(x).$$

Let

$$\begin{aligned}
\gamma^{(0)}(z) &= (1-z^2)\int_{-1}^{1} \frac{1}{1-2\sqrt{q}xz+qz^2} d\sigma(x), \\
\gamma^{(1)}(z) &= (1-z^2)\int_{1<|x|\le\varrho} \frac{1}{1-2\sqrt{q}xz+qz^2}\, d\sigma(x).
\end{aligned}$$

For $-1 \le x \le 1$ the expression $1-2\sqrt{q}xz+qz^2$ vanishes only on the circle of radius $q^{-1/2}$. Thus $\gamma^{(0)}(z)$ has the desired property. For $1 < |x| \le \varrho$ the expression $1-2\sqrt{q}xz+qz^2$ vanishes only on the intervals

$$\left[-\frac{\varrho+\sqrt{\varrho^2-1}}{\sqrt{q}}, -\frac{\varrho-\sqrt{\varrho^2-1}}{\sqrt{q}}\right], \quad \left[\frac{\varrho-\sqrt{\varrho^2-1}}{\sqrt{q}}, \frac{\varrho+\sqrt{\varrho^2-1}}{\sqrt{q}}\right].$$

By (11) we have that $\gamma = q^{1/2}(\varrho+\sqrt{\varrho^2-1})$. This shows that $\gamma^{(1)}$ is analytic where it has been required.

The functional equation follows immediately from the formula

$$\frac{z\gamma^{(1)}(z)}{1-z^2} = \int_{1<|x|\le\varrho} \frac{1}{z^{-1}-2\sqrt{q}x+qz}\, d\sigma(x).$$

□

Remark. Combining (6) and (10) yields

$$\gamma_{2n} = q^n\left\{\int_{\varrho_0}^{\varrho}(x+\sqrt{x^2-1})^{2n}\frac{(x+\sqrt{x^2-1})^2-q^{-1}}{2\sqrt{x^2-1}(x+\sqrt{x^2-1})}\, d\widetilde{\sigma}(x)+o(1)\right\}.$$

We have

$$\begin{aligned}
h(\varrho_0) := \frac{(\varrho_0+\sqrt{\varrho_0^2-1})^2-q^{-1}}{2\sqrt{\varrho_0^2-1}(\varrho_0+\sqrt{\varrho_0^2-1})} &\ge \frac{(x+\sqrt{x^2-1})^2-q^{-1}}{2\sqrt{x^2-1}(x+\sqrt{x^2-1})}, \\
\frac{(\varrho+\sqrt{\varrho^2-1})}{\varrho}x &\ge x+\sqrt{x^2-1}.
\end{aligned}$$

Therefore, in view of (2), we get

$$\begin{aligned}\gamma_{2n} &\le q^n \left\{ h(\varrho_0) \frac{(\varrho+\sqrt{\varrho^2-1})^{2n}}{\varrho^{2n}} \int_0^{\varrho} x^{2n} d\widetilde{\sigma}(x) + o(1) \right\} \\ &= q^n h(\varrho_0) \left\{ (\varrho+\sqrt{\varrho^2-1})^{2n} \frac{\mu^{*2n}(e)}{\varrho^{2n}} + o(1) \right\}.\end{aligned}$$

Finally we obtain

$$\frac{\gamma_{2n}}{\gamma^{2n}} \frac{\varrho^{2n}}{\mu^{*2n}(e)} = \frac{\gamma_{2n}}{\mu^{*2n}(e)} \left\{ \frac{\varrho}{\sqrt{q}(\varrho+\sqrt{\varrho^2-1})} \right\}^{2n} \le h(\varrho_0) + o(1).$$

We conjecture that the opposite estimate also holds; i.e. the quantity on the left hand side is bounded away from zero, by a positive constant depending only on ϱ. This conjecture can be checked easily if the measure σ is smooth in the neighbourhood of ϱ and the density has zero of finite order at ϱ.

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If $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is analytic in the complex plane except the half line $[1, +\infty)$, then the ratio a_{n+1}/a_n converges to 1.

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